TECHNICAL UNIVERSITY OF MOMBASA

FACULTY OF APPLIED AND HEALTH SCIENCES

DEPARTMENT OF MATHEMATICS AND PHYSICS

UNIVERSITY EXAMINATION FOR:

MASTER OF SCIENCE IN APPLIED STATISTICS

AMA 5109: STOCHASTIC PROCESSES

END OF SEMESTER EXAMINATION

SERIES: AUGUST **TIME:** **3** HOURS

DATE: AUGUST 2019

INSTRUCTIONS:

This Exam consists of Five Questions.

Answer **QUESTION ONE (Compulsory)** and any other TWO from the remaining questions

QUESTION ONE [30 MARKS]

a) Explain the following terms as used in stochastic processes

i. Independent increments [2 Marks]

ii. Stationary increments [2 Marks]

b) Using the probability generating function find the mean and variance of a distribution defined by $p_r\{X = k\} = pq^k$ for p+q = 1 and k = 0,1,2,…

[6 marks]

c) Let $N(t)$ be a Poisson process with intensity $\lambda = 2$, and let $X_1, X_2, \cdots$ be the corresponding interarrival times.

i. Find the probability that the first arrival occurs after $t = 0.5$, i.e., $P(X_1 > 0.5)$. [1 mark]

ii. Given that we have had no arrivals before $t = 1$, find $P(X_1 > 3)$.

[1 mark]

iii. Given that the third arrival occurred at time $t = 2$, find the probability that the fourth arrival occurs after $t = 4$. [2 marks]

iv. I start watching the process at time $t = 10$. Let T be the time of the first arrival that I see. In other words, T is the first arrival after $t = 10$. Find $E(T)\ and\ Var(T)$. [2 marks]

d) In a certain process , the probability of n offspring from one ancestor is geometric with probability p

i) Find the range of values for which the process will die out with probability one. [3 marks]

ii) For p outside this range, find the probability of extinction [2 marks]

iii) If p is chosen so that the probability of a process never dies out is 0.999, what is the probability that an individual will have no offspring

[1 marks]

e) The number of failures $N(t)$, which occur in a computer network over the time interval $[0;\ t)$, can be described by a homogeneous Poisson process $\{\ N(t), t \geq 0\ \}$. On an average, there is a failure after every 4 hours, i.e. the intensity of the process is equal $\lambda = 0.25/hr$

i) What is the probability of at most 1 failure in [0; 8), at least 2 failures in [8; 16), and at most 1 failure in [16; 24) (time unit: hour)? [4 marks]

ii) What is the probability that the third failure occurs after 8 hours?

[4 marks]

QUESTION TWO [15 MARKS]

a) Let $\{\ N(t), t \geq 0\ \}$ be a counting process. State four conditions that this process needs to satisfy for it to be a poisson process [4 marks]

b) Let $\{\ N(t), t \geq 0\ \}$, be a Poisson process and $p_n(t + h) = p[N(t + h) = n]$. Derive the distribution of $N(t)$. [8 marks]

c) Let $S = \sum_{i=1}^{n} X_i$ denote the waiting time until the n^{th} event has occurred in a Poisson process, where the X_i are the interarrival times. Show that $S \sim gamma(n, \lambda)$ [8 marks]

QUESTION THREE [20 MARKS]

a) The National bank Kilifi branch has a single cashier. During the rush hours, customers arrive at the rate of 10 per hour. The average number of customers that can be processed by the cashier is 12 per hour. On the basis of this information, find the following:

i) Probability that the cashier is idle [2 marks]

ii) Average number of customers in the queuing system [2 marks]

iii) Average time a customer spends in the system [2 marks]

iv) Average number of customers in the queue [2 marks]

v) Average time a customer spends in the queue [2 marks]

b) Show that in a poison process with mean arrival rate of λ per unit time, the interarrival times follow an exponential distribution [5 marks]

c) Let $W(t)$ be a standard Brownian motion, and $0 \leq s < t$. Find the conditional PDF of $W(s)$ given $W(t) = a$. [5 marks]

QUESTION FOUR [15 MARKS]

a) Define the following terms as used with Markov Chains

i) Periodic State [2 mark]

ii) Ergodic State [2 mark]

iii) Absorbing state [2 mark]

b) Let p_{jk}^{n} be the probability of moving from E_j to E_k in n steps regardless of the number of entrances into E_k prior to n and f_{jk}^{n} be the probability of entering E_k from E_j in n steps for the first time , show that there exits a relationship between these probabilities given by $P(s) = \frac{1}{1 - F(s)}$ [4 marks]

c) Example: Suppose we are tracking smokers and non-smokers: S = smokers, N = non-smokers. Each year 10% of smokers stop smoking and 5% of non-smokers start smoking.

i. Draw the transition diagram for this process [2 marks]

ii. If initially 40% of the population smokes, track the trend over 4 years. [4 marks]

iii. Find the stable state probabilities [4 marks]

QUESTION FIVE [20 MARKS]

a) Consider a System with state space consisting of the integers $i = 0, \pm 1, \pm 2, \ldots$ and has transition probabilities given by $p_{i,i+1} = p$ and $p_{i,i-1} = 1 - p$, $i = 0, \pm 1, \pm 2, \ldots$ where $0 < p < 1$. Show that when $p = \frac{1}{2}$, the states are recurrent while they are transient when $p \neq \frac{1}{2}$ [8 Marks]

b) Let $N(t), t \geq 0$ be a Poisson process with rate λ. Let$0 < s < t$. Show that given $N(t) = n,\;\; N(s)$ is a binomial random variable with parameters n and $p = \frac{s}{t}$

[4 marks]

c) Consider the Markov chain with the following transition matrix

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.3333 & 0.6667 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

i. Is this chain irreducible? Explain [2 Marks]

ii. Is this chain aperiodic? Explain [2 marks]

iii. Find the stationary distribution for this chain. [4 marks]