AA

TECHNICAL UNIVERSITY OF MOMBASA

FACULTY OF APPLIED AND HEALTH SCIENCES

DEPARTMENT OF MATHEMATICS & PHYSICS

UNIVERSITY EXAMINATION FOR:

BMCS YEAR IV SEMESTER I

AMA 4414: FIELD THEORY

SPECIAL/ SUPPLIMENTARY EXAMINATIONS

SERIES: September 2018

TIME: 2HOURS

DATE: September 2018

ANSWER QUESTION ONE AND ANY OTHER TWO QUESTIONS

QUESTION ONE (30 MARKS)

a)

(i) Define a conjugate element (2 Marks)

(ii) Find all the conjugate of each of the given elements over the given fields

1. $3 + \sqrt{2}$ over $\mathbb{Q}$ (2 Marks)
2. $\sqrt{2} + i$ over $\mathbb{R}$ (2 Marks)
3. $\sqrt{1 + \sqrt{2}}$ over $\mathbb{Q}\sqrt{2}$ (2 Marks)

b)

(i) What is a primitive n^{th} root of unity? (2 Marks)

(ii) Find the primitive 5^{th} root of unity in $\mathbb{Z}_{11}$ (6 Marks)

c) For each of the given polynomial in $\mathbb{Q}[x]$. Find the degree over $\mathbb{Q}$ of the splitting field over $\mathbb{Q}$ of the polynomial;

(i) $x^2 + 3$ (3 Marks)

(ii) $x^3 - 1$ (4 Marks)

d)

(i) Define a zero divisor (1 Mark)

(ii) Outline the zero divisors in $\mathbb{Z}_6$ (1 Mark)

e) Prove that if $f \in \mathbb{Z}[x]$ is monic, then every monic factor of f is in $\mathbb{Q}[x]$ lies in $\mathbb{Z}[x]$

(5 Marks)

QUESTION TWO (20 MARKS)

a) State and prove Eisenstein's criterion (7 Marks)

b)

(i) Define an algebraic extension (2 Marks)

(ii) Prove that a field extension E/F is finite if and only if E is algebraic and finitely generated (as a field) over F. (7 Marks)

c) Prove that if P is prime, then $x^{p-1} + \cdots + 1$ is irreducible: hence $\mathbb{Q}\left[e^{\frac{2\pi i}{p}}\right]$ has a degree $p - 1$ over $\mathbb{Q}$ (4 Marks)

QUESTION THREE (20 MARKS)

a) Prove that $\propto = \sum \frac{1}{2^{n!}}$ is transcendental (7 Marks)

b) Prove that if E is a field extension of F, we may view E as a vector space over F.

(7 Marks)

c) For each of the following $a \in \mathbb{C}$, show that $\propto$ is algebraic over F.

(i) $1 + \sqrt{2}$ (3 Marks)

(ii) $1 + i$ (3 Marks)

QUESTION FOUR (20 MARKS)

a)

(i) Define a splitting polynomial (3 Marks)

(ii) Let E be an extension field of F and $\propto \in E$. State what is meant by $\propto$ is algebraic over F. (2 Marks)

b) Prove that if E is a finite field of characteristic p, then E contains exactly p^n elements for $n \in \mathbb{Z}^+$. (4 Marks)

c) Find the degree over $\mathbb{Q}$ and the splitting fields of;

(i) $x^4 - 1$ (3 Marks)

(ii) $(x^2 - 2)(x^2 - 3)$ (3 Marks)

d) Find the degree and basis of;

(i) $\mathbb{Q}\sqrt{2}$ over $\mathbb{Q}$ (2 Marks)

(ii) $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ over $\mathbb{Q}$ (3 Marks)

QUESTION FIVE (20 MARKS)

a Given the degree of each of the following

(i) $\mathbb{Q}(\sqrt{2}, \sqrt{6}+\sqrt{10})$ over $\mathbb{Q}(\sqrt{3}+\sqrt{5})$ (1 Mark)

(ii) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ (1 Mark)

(iii) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}(\sqrt{3})$ (1 Mark)

b) Which of the following is a field?

(i) $F = \{a + bi \mid a, b \in \mathbb{Q}\}$ and $i = \sqrt{-1}$ (4 Marks)

(ii) $F = \{a + bi \mid a, b \in \mathbb{Z}\}$ and $i = \sqrt{-1}$ (3 Marks)

c) State and prove the Euclid's algorithm (10 Marks)