TECHNICAL UNIVERSITY OF MOMBASA

A Centre of Excellence

Faculty of Applied & Health Sciences

PARTIAL DIFFERENTIAL EQUATIONS II TIME ALLOWED: 2HOURS

PAPER A

Instructions to Candidates:
You should have the following for this examination
- *Answer Booklet*
- *Scientific Calculator*

This paper consists of **FIVE** questions and **TWO** sections **A** and **B.**
Answer question **ONE** (**COMPULSORY)** and any other **TWO** questions
Maximum marks for each part of a question are as shown
This paper consists of **THREE** printed pages.

SECTION A (COMPULSORY)

Question ONE (30 marks)

a. Obtain the solution of the following initial value problem $u_{xx} = 4xy + e^x$ with the initial condition $u(0, y) = y$, $u_x(0, y) = 1$ (5 marks)

b. Show that the Laplace's equation $\nabla^2 u = 0$ is satisfied by the function $u = \frac{1}{r}$

where $$u = \frac{1}{\left[(x - x_o)^2 + (y - y_0)^2 + (z - z_0)^2\right]^{\frac{1}{2}}}$$ (6 marks)

c. Consider the following second order partial differential equation:-

$$x^2 u_{xx} - 2xy\, u_{xy} + y^2 u_{yy} = e^x$$

(i) Classify it. (2 marks)

(ii) Reduce to canonical form. (7 marks)

(iii) Find the general solution in terms of arbitrary functions. (2 marks)

d. Use the method of separation of variables to solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$

given $u(x,0) = 8e^{-4x}$. (8 marks)

SECTION B

Question TWO (20 marks)

a. Show that if Laplace's equation $\nabla^2 u = 0$ in Cartesian coordinate is transformed by introducing plane polar coordinates (r,θ), defined by the relation $x = r\cos\theta$, $y = r\sin\theta$ it takes the form $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ (10 marks)

b. Solve the boundary value problem for a rectangle defined by Laplace's equation
PDE: $\nabla^2 u = 0$, $0 \le x \le a$, $0 \le y \le b$ with the following boundary conditions
BC's: $u\,(x,0) = u(a,y) = 0$, $u\,(0,y) = 0$, $u\,(x,b) = 0$, $u\,(x,0) = f\,(x)$ (1 0 marks)

Question THREE (20 marks)

a. A rod of length l with insulated side is initially at a uniform temperature u_o. Its ends are suddenly cooled to 0° and are kept at that temperature.

i. Find the temperature function of this problem (2 marks)

ii. Set up the initial and boundary conditions of the temperature function given in (i) above. (3 marks)

iii. Solve the temperature function subject to the initial and boundary conditions in (i) and (ii) above (15 marks)

Question FOUR (20 marks)

a. Show that $u(x,t) = 2^{-8t}\sin 2x$ is a solution to the boundary value problem

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2},\ u(0,t) = u(\pi,t) = 0,\ u(x,0) = \sin 2x$$ (7 marks)

b. An infinitely long string having one end at initially at rest on the x-axis . At $t=0$ the end $x=0$ begins to move along the u-axis in a manner described by $u(0,t)=a\cos\sigma t$.

(a) State the PDE for the one dimensional wave equation of this problem. Show this with an illustration of a sketch diagram. (2 marks)

(b) Using Laplace transform method, find the displacement $u(x,t)$ of the string at any point at any time subject to the boundary conditions and initial conditions given as

B.C $\quad u(0,t)=a\cos\sigma t$, (i)

$u(x,t)$ bounded as $t\to\infty$. (ii)

I.C $\quad u(x,0)=0$ (iii)

$u_t(x,0)=0$ (iv) (11 marks)

Question FIVE (20 marks)

Using the method of separation of variables, Solve the Neumann problem for a rectangle defined with the following initial and boundary conditions as follows :- (20 marks)

PDE: $\nabla^2 u=0$, $\quad 0\le x\le a,\ 0\le y\le b$

BCs: $u_x(0,y)=u_x(a,y)=0$, $\quad u_y(x,0)=0$, $\quad u_y(x,b)=f(x)$