TECHNICAL UNIVERSITY OF MOMBASA

AMA 5204 STOCHASTIC PROCESSES

INSTRUCTIONS: ANSWER ANY THREE QUESTIONS

QUESTION ONE [20 MARKS]

a) Define the following terms as used with Markov Chains
   i) Periodic State [1 mark]
   ii) Ergodic State [1 mark]
   iii) Absorbing state [1 mark]

b) Let p_{jk}^{n} be the probability of moving from E_j to E_k in n steps regardless of the number of entrances into E_k prior to n and f_{jk}^{n} be the probability of entering E_k from E_j in n steps for the first time , show that there exits a relationship between these probabilities given by $P(s) = \frac{1}{1-F(s)}$ [4 marks]

c) Classify the states for this infinite Markov Chain [8 marks]

$$\begin{bmatrix} & E_1 & E_2 & E_3 & E_4 & E_5 & \cdots \\ E_1 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & \cdots \\ E_2 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & \cdots \\ E_3 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 & \cdots \\ E_4 & \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} & \cdots \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

QUESTION TWO [20 MARKS]

a) Suppose that $p_j = p_r(z = j)$ forms a geometric series $p_j = br^{j-1}$ j = 1,2,...

   where 0 < r < 1 and 0 < b <1-r while $p_0 = 1 - \sum_{j=1}^{\infty} p_j$

   i) Find the corresponding p.g.f and the mean [5 marks]

   ii) Show that the equation $s = p(s)$ has its only positive roots 1 and

   $$s = \frac{1-(r+b)}{r(1-r)}$$ [5 marks]

b) Consider a series of Bernouli trials with probability of success p. Suppose that

X denotes the number of failures following the first success and Y the number of failures following the first success and preceding the second success

i) Using the bivariate p.g.f obtain Variance of X and the Variance of Y [6 marks]

ii) Show that the X and Y are independent [4 marks]

QUESTION THREE[20 MARKS]

Consider a process whose difference equation is given by

$p_n^{/}(t) = -{}^{a}\, p_n(t) + {}^{a}\, p_{n-1}(t) \qquad n \geq 1$ and $p_o^{'}(t) = -{}^{a}\, p_o(t) \qquad n = 0$. Suppose the initial condition are $p_n(0) = 1$ for n = 0 and 0 otherwise ,

a) Obtain the probability generating function of this process [9marks]

b) What is the probability that the population is of size n at time t i.e $p_n(t)$ [5 marks]

c) Find the expected value and the variance of n . i.e $p_n(t)$ [6marks]

QUESTION FOUR [20 MARKS]

a) Using the probability generating function find the mean and variance of a distribution defined by $p_r\{X = k\} = pq^k$ for p+q = 1 and k = 0,1,2,… [6 marks]

b) A certain kind of nuclear particle splits into 0,1 or 2 particles with probability $\frac{1}{4}, \frac{1}{2}$ and $\frac{1}{4}$ respectively and then dies. The individual particles act independently of each other. Given a particle, let z_1, z_2 and z_3 denote the number of particles in the first, second and third generations . Find (i) $p_r[z_2 > 0]$ (ii) $p_r[z_3 = 0]$ [7 marks]

c) In a certain process , the probability of n offspring from one ancestor is geometric with probability p

i) Find the range of values for which the process will die out with probability one. [3 marks]

ii) For p outside this range, find the probability of extinction [2 marks]

iii) If p is chosen so that the probability of a process never dies out is 0.999, what is the probability that an individual will have no offspring [2 marks]

QUESTION FIVE [20 MARKS]

a) Let $X_i, i = 1, 2, \ldots$ be identically and independently distributed random variables with $\mathrm{P}\left\{\mathrm{X_i = k}\right\} = \mathrm{p_k}$ and p.g.f , $p(s) = \sum_k p_k S^k$ for $i = 1, 2, \ldots$. Let $S_N = X_1 + X_2 + \ldots + X_N$ where N is a random variable independent of the X_i's.

Let the distribution of N be given by $\Pr\{N = n\} = g_n$ and the p.g.f of N be $G(s) = \sum_n g_n S^n$, Show that the p.g.f H(s) of S_N is given by

$H(s) = \sum_j p_r\{S_N = j\} S^j = G[p(s)]$ [10 marks]

b) Let X have a p.d.f $\mathrm{P}\left\{\mathrm{X = k}\right\} = \mathrm{p_k}, k = 0, 1, 2, \ldots$ with p.g.f $p(s) = \sum_{k=0}^{\infty} p_k S^k$

and ¤$(s) = \sum_{k=0}^{\infty} q_k S^k$. Show that ¤$(s) = \frac{1 - p(s)}{1 - s}$ if $q_k = p_r\{X > k\}$ and

¤$(s) = \frac{p(s)}{1 - s}$ if $q_k = p_r\{X \le k\}$ for k = 0,1,2,...

[10 marks]