TECHNICAL UNIVERSITY OF MOMBASA

AMA 5204 STOCHASTIC PROCESSES

INSTRUCTIONS: ANSWER ANY THREE QUESTIONS

QUESTION ONE [20 MARKS]

a) Using the probability generating function find the mean and variance of a distribution defined by $p_r\{X = k\} = pq^k$ for p+q = 1 and k = 0,1,2,...

[4marks]

b) Define the following terms as used in markov chains

i.

c) Consider a series of Bernouli trials with probability of success p. Suppose that X denotes the number of failures following the first success and Y the number of failures following the first success and preceding the second success

i) Using the bivariate p.g.f obtain Variance of X and the Variance of Y

[6 marks]

ii) Show that the X and Y are independent [4 marks]

QUESTION TWO [20 MARKS]

a) Let $X_i, i = 1,2,...$ be identically and independently distributed random variables with $\mathrm{P}\{\mathrm{X_i} = \mathrm{k}\} = \mathrm{p_k}$ and p.g.f , $p(s) = \sum_k p_k S^k$ for $i = 1,2,...$. Let $S_N = X_1 + X_2 + ... + X_N$ where N is a random variable independent of the X_i's.

Let the distribution of N be given by $\Pr\{N = n\} = g_n$ and the p.g.f of N be $G(s) = \sum_n g_n S^n$, Show that the p.g.f H(s) of S_N is given by

$$H(s) = \sum_j p_r\{S_N = j\} S^j = G[p(s)]$$ [7 marks]

b) Let X have a p.d.f $\mathrm{P}\{\mathrm{X} = \mathrm{k}\} = \mathrm{p_k}, k = 0,1,2,...$ with p.g.f $p(s) = \sum_{k=0}^{\infty} p_k S^k$ and

$¤(s) = \sum_{k=0}^{\infty} q_k S^k$. Show that $¤(s) = \dfrac{1 - p(s)}{1 - s}$ if $q_k = p_r\{X > k\}$ and $¤(s) = \dfrac{p(s)}{1 - s}$

if $q_k = p_r\{X \le k\}$ for k = 0,1,2,...

[8 marks]

QUESTION THREE[20 MARKS]

a) Consider a process whose difference equation is given by

$p_n^{/}(t) = -^{a}\, p_n(t) + ^{a}\, p_{n-1}(t) \qquad n \geq 1$ and $p_o^{'}(t) = -^{a}\, p_o(t) \qquad n = 0$. Suppose the initial condition are $p_n(0) = 1$ for n = 0 and 0 otherwise ,

i. Obtain the probability generating function of this process [7 marks]

ii. What is the probability that the population is of size n at time t i.e $p_n(t)$ [3 marks]

iii. Find the expected value and the variance of n . i.e $p_n(t)$ [5 marks]

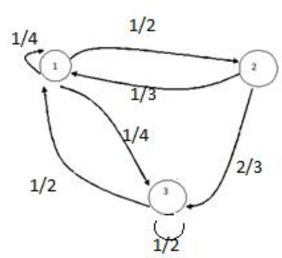
b) Let $N(t)$ be a Poisson process with rate ^{a} . Consider a very short interval in length ¢ show that the number of arrivals in the in this interval has the property that

$P(N(¢) \geq 2) = o(¢)$ [5 marks]

QUESTION FOUR [20 MARKS]

a) Consider the markov Chain shown in the figure below. Assume $X_o = 1$, and let R be the first time that the chain returns to state 1 i.e. $R = \min\{n \geq 1 : X_n = 1\}$. Find

$E[R \mid X_o = 1]$ [2 marks]



b) Let $N(t)$, $t \in [0, \infty)$ be a Poisson process with rate $^{a} = 0.5$ Find the probability that

i. There is no arrival in (3,5] [1 marks]

ii. There is exactly one arrival in each of the following intervals: (0,1], (1,2], (2,3] and (3,4] [1 marks]

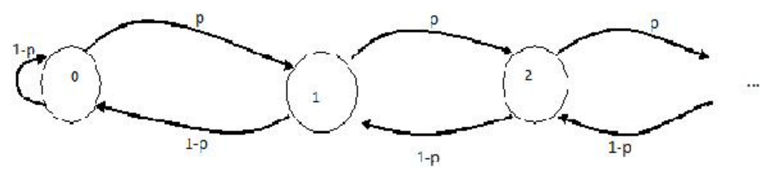
c) Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rate $^{a}{}_1 = 1$ and $^{a}{}_2 = 2$, respectively. Let $N(t)$ be the merged process $N(t) = N_1(t) + N_2(t)$

i. Find the probability that $N(1) = 2$ and $N(2) = 5$ [2 marks]

ii. Given that $N(1) = 2$, find the probability that $N_1(1) = 1$ [3 marks]

d) Consider a markov Chain shown in the figure below. Assume that 0.5<p<1. Does this Chain have a limiting distribution?

Find $\lim_{n \to \infty} P(X_n = j \mid X_o = i)$, for $j \in \{0,1,2,...\}$ [6marks]



e) An automobile insurance company places its policy holders into one of two categories when the policy renews;low risk or high risk. A motorist is high risk if the policy holder has received a moving violation (ticket) within the past 12 months and low risk having had zero tickets in the past 12 months. Based on the government data it is estimated that any driver chosen at random has a 10% chance of having had a moving violation within the past twelve months. Based on company data , a current policy holder that is at high risk has a 60% chance of being denoted high risk again when the policy renews and a 40% chance of being moved to low risk. A low risk driver on the other hand has a 15% chance of moving to the high risk category and 85% chance of remaining at low risk. If all of its new policy holders this year are representative of the driving population in general, what will be the company's high risk distribution

i) One year from now [2 marks]

ii) Two years from now. [2 marks]

iii) Many years from now [2 marks]